D1-82-0319

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# A "String Algorithm" for Shortest Paths in Directed Networks



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Mathematics Research

November 1963

D1-82-0319

# A "STRING ALGORITHM" FOR SHORTEST PATHS IN DIRECTED NETWORKS

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Mathematical Note No. 333

Mathematics Research Laboratory

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November 1963

The literature contains several algorithms for finding the shortest path between two nodes P and Q of a network, where the distances or arc-lengths are assumed to be positive. (For references, consult the review article by Pollack and Weibenson [3] and the book by Ford and Fulkerson [1].) Some of the algorithms, and in particular some of the analogue devices, are applicable only when the distance matrix is symmetric. As was remarked in [1] and [3], this is true of the simplest of the analogue procedures ----- the "string algorithm" reported by Minty [2]. It consists of making an inelastic string model of the network, with knots corresponding to nodes and string-lengths proportional to the corresponding distances, and then stretching the knots P and Q as far apart as is possible without breaking the string; this produces at least one straight path from P to Q, and each such straight path corresponds to a shortest path in the network.

Networks with asymmetric distance matrices are most conveniently represented by means of directed networks, in which every arc is regarded as a one-way street of the appropriate length. In the present note we describe a simple cutting procedure (related to one suggested by Thomas Seidman) which can be combined with any algorithm for undirected networks (symmetric distance matrix) so as to form a shortest-path algorithm for directed networks (asymmetric distance matrix). In particular, the cutting and stretching can be alternated to form a "string algorithm" for directed networks.

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For each directed network $\underset{\sim}{N}$, let $\underset{\sim}{N}^u$ denote the corresponding undirected network.

THEOREM *Suppose that* $P$ *and* $Q$ *are nodes of a finite directed network* $\mathbf{N}_1$ *which has* $v$ *nodes and* $e$ *arcs, and that there is a path from* $P$ *to* $Q$ *in* $\mathbf{N}_1$. *Suppose that* $\mathbf{A}$ *is an algorithm for finding shortest paths in undirected networks, and let the sequential procedure* $\mathbf{S}_1, \mathbf{C}_1, \mathbf{S}_2, \mathbf{C}_2, \ldots$ *be as follows:*

$\mathbf{S}_i$) *Apply* $\mathbf{A}$ *to the undirected network* $\mathbf{N}_i^u$ *to find a shortest path* $\pi_i$ *from* $P$ *to* $Q$ *in* $\mathbf{N}_i^u$. *Suppose* $\pi_i$ *is given by*

$$V_0^i \alpha_1^i V_1^i \alpha_2^i V_2^i \ldots V_{h(i)-1}^i \alpha_{h(i)}^i V_{h(i)}^i,$$

*where the arcs* $\alpha_j^i$ *and the nodes* $V_j^i$ *are listed as they appear in traversing* $\pi_i$ *from* $P = V_0^i$ *to* $Q = V_{h(i)}^i$.

$\mathbf{C}_i$) *If* $\pi_i$ *is also a path in* $\mathbf{N}_i$, *terminate the procedure. If* $\pi_i$ *is not a path in* $\mathbf{N}_i$, *there exists a smallest index* $r(i)$ *and a largest index* $s(i)$ *(possibly the same) such that the directions of* $\alpha_{r(i)}^i$ *and* $\alpha_{s(i)}^i$ *in* $\pi_i$ *are opposite to their directions in* $\mathbf{N}_i$. *Let* $\mathbf{N}_{i+1}$ *be the directed network that is obtained from* $\mathbf{N}_i$ *by deleting every arc of* $\mathbf{N}_i$ *that (like* $\alpha_{r(i)}^i$*) ends in* $\mathbf{N}_i$ *at* $V_{r(i)-1}^i$ *but is not* $\alpha_{r(i)-1}^i$, *and deleting every arc of* $\mathbf{N}_i$ *that (like* $\alpha_{s(i)}^i$*) starts in* $\mathbf{N}_i$ *at* $V_{s(i)}^i$ *but is not* $\alpha_{s(i)+1}^i$. *There is a path* $\Sigma_i$ *from* $P$ *to* $Q$ *in* $\mathbf{N}_{i+1}$ *such that* $\Sigma_i$ *is actually a shortest path from* $P$ *to* $Q$ *in* $\mathbf{N}_i$.

*The procedure terminates at some stage* $\mathbf{C}_t$ *for which*

$$t \leq \min(v, (e+2)/2),$$

*and the path* $\pi_t$ *is a shortest path from* $P$ *to* $Q$ *in the directed network* $\mathbf{N}_i$.

(*The same conclusion holds if* $\mathbf{C}_i$ *requires only the first of the two deletions specified above, or if it requires only the second.*)

Proof. Of course the algorithm itself does not involve the actual *construction* of $\Sigma_i$, but we require the existence of $\Sigma_i$ (when $\mathbf{C}_i$ does not specify termination) to show that the sequential procedure $\mathbf{S}_1$, $\mathbf{C}_1$, $\mathbf{S}_2$, $\mathbf{C}_2$, ... can actually be followed and that each of the paths $\Sigma_1, \Sigma_2, \ldots$ is a shortest path from P to Q in $\mathbf{N}_1$. Since $\mathbf{N}_1$ is finite, the procedure must terminate at some stage $\mathbf{C}_t$ and then $\pi_t$ is a shortest path from P to Q in $\mathbf{N}_i$.

Suppose $\mathbf{C}_i$ does not specify termination and let $\Sigma_{i-1}$ be a shortest path from P to Q in $\mathbf{N}_i$, given by

$$W_0^i\ \beta_1^i\ W_1^i\ \beta_2^i\ W_2^i\ \ldots\ W_{l(i)-1}^i\ \beta_{l(i)}^i\ W_{l(i)}^i,$$

where of course $W_0^i = P$ and $W_{l(i)}^i = Q$. In constructing $\Sigma_i$, we consider the following three possibilities:

(i) no $W_j^i$ is equal to either $V_{r(i)-1}^i$ or $V_{s(i)}^i$;

(ii) there exists $j$ such that $W_j^i = V_{r(i)-1}^i$ and $j < k \Rightarrow W_k^i \neq V_{s(i)}^i$;

(iii) there exist $j$ and $k$ such that $j < k$, $W_j^i = V_{r(i)-1}^i$ and $W_k^i = V_{s(i)}^i$.

When (i) holds, we define $\Sigma_i = \Sigma_{i-1}$. When (ii) holds, we obtain $\Sigma_i$ by following $\pi_i$ from P to $V_{m(i)-1}^i$ and then following $\Sigma_{i-1}$ from $V_{m(i)-1}^i$ to Q. When (iii) holds, we obtain $\Sigma_i$ by following $\pi_i$ from P to $V_{m(i)-1}^i$, next following $\Sigma_{i-1}$ from $V_{m(i)-1}^i$ to $V_{n(i)}^i$, and then following

$\pi_i$ from $V^i_{n(i)}$ to $Q$. In each case, it is easily verified that $\Sigma_i$ has the stated properties. Thus the existence of $t$ is established and it remains only to show that $t \leq \min(v,(e+2)/2)$.

Let us review the special properties of certain nodes and arcs of $\underset{\sim}{N}_i$ relative to $\underset{\sim}{N}_i$ itself and relative to $\underset{\sim}{N}_j$ for $j > i$.

(a) $V^i_{r(i)-1} \neq Q$. If $V^i_{r(i)-1} = P$, then at least one arc of $\underset{\sim}{N}_i$ ends at $V^i_{r(i)-1}$ but no arc of $\underset{\sim}{N}_j$ ends there. If $V^i_{r(i)-1} \neq P$, then at least two arcs of $\underset{\sim}{N}_i$ end at $V^i_{r(i)-1}$ but at most one arc of $\underset{\sim}{N}_j$ endsthere.

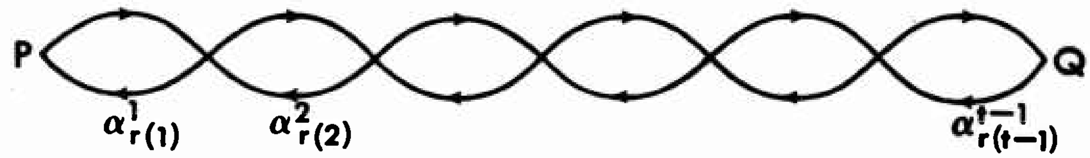
(b) $\alpha^i_{r(i)}$ ends at $P$ or is coterminal with another arc of $\underset{\sim}{N}_i$; $\alpha^i_{r(i)}$ does not end at $Q$ and does not appear in $\underset{\sim}{N}_j$.

(c) $\alpha^i_{r(i)-1}$ does not end at $P$ or $Q$, and is nonexistent if $\alpha^i_{r(i)}$ ends at $P$. If $\alpha^i_{r(i)}$ does not end at $P$, then $\alpha^i_{r(i)-1}$ is coterminal with another arc of $\underset{\sim}{N}_i$ but not with another arc of $\underset{\sim}{N}_j$.

We see from (a) that the $t$ nodes $Q, V^1_{r(1)-1}, V^2_{r(2)-1}, \ldots, V^{t-1}_{r(t-1)-1}$ are pairwise distinct, and consequently $t \geq v$. From (b) and (c) it follows that the arcs $\alpha^i_{r(1)-1}, \alpha^1_{r(1)}, \alpha^2_{r(2)-1}, \alpha^2_{r(2)}, \ldots, \alpha^{t-1}_{r(t-1)-1}, \alpha^{t-1}_{r(t-1)}$ are pairwise distinct. If $\alpha^i_{r(i)}$ ends at $P$, then $\alpha^i_{r(i)-1}$ does not appear, but this happens for at most one value of $i$, and since at least one arc of $\underset{\sim}{N}_1$ ends at $Q$ we conclude that $e \geq 2t - 2$.

The above reasoning completes the proof when $\underset{\sim}{C}_i$ is as originally described, and also when $\underset{\sim}{C}_i$ is replaced by $\underset{\sim}{C}_i'$ which requires only the first of the specified deletions. Similar reasoning applied to $\underset{\sim}{C}_i''$, which

Figure 1

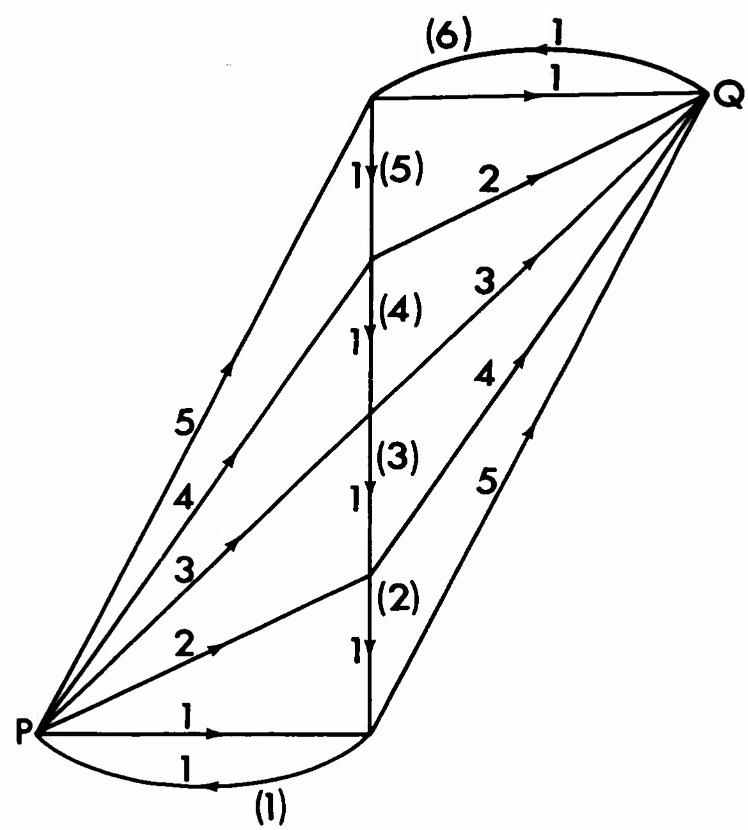



v nodes. 2(v-1) arcs, all of the same length.

$t = v = (e+2)/2$

($\pi_i$ follows upper arcs except at $\alpha^i_{r(i)}$)

Figure 2

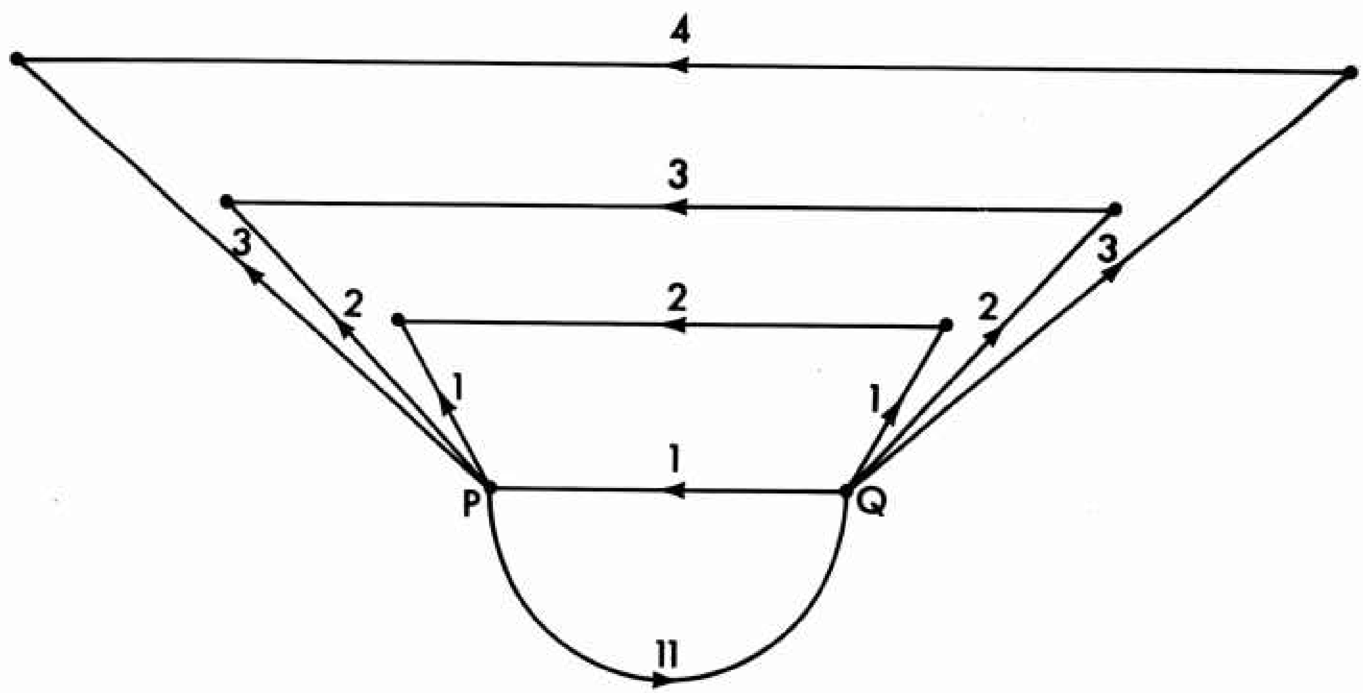



v nodes. 3v - 5 arcs, with lengths as indicated.

$\alpha^i_{r(i)}$(= $\alpha^i_{s(i)}$) indicated by (i)

$t = v$

Figure 3




2n nodes.

3n - 1 arcs.

$$t = n + 1 = \frac{v+2}{2} = \frac{e+4}{3}$$

($\pi_i$ is uniquely determined)

## REFERENCES

1. L. R. Ford, Jr. and D. R. Fulkerson, Flows in Networks, Princeton University Press, 1962.

2. G. J. Minty, "A Comment on the Shortest-Route Problem," Opns. Res. 5, 724 (1957).

3. M. Pollack and W. Wiebenson, "Solutions of the Shortest-Route Problem-- A Review," Opns. Res. 8, 224-230 (1959).